Abstract versions of Korovkin theorems on modular spaces via statistical relative summation process for double sequences

Sevda Yıldız

Department of Mathematics, Sinop University, Sinop, Turkey
E-mail: sevdaorhan@sinop.edu.tr

Abstract

In this paper, we studied the abstract versions of Korovkin type approximation theorems via statistical relative $\mathcal{A}-$Summation process in modular spaces for double sequences. Then, we discuss the results which are obtained by special choice of the scale function and the matrix sequences and we give an application that shows our results are stronger than studied before. Finally, we study an extension to non-positive linear operators.

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1 Introduction

The classical Korovkin type theorem deals with the approximation only using test functions that provides the approximation in whole space[19]. Several authors work on this theorem and this theorem has been extended with the use of summability methods since they make a non-convergent sequence convergent (see [1, 11, 14, 15, 31, 34]). One of the most important papers on Korovkin type approximation theory is [5] in which the authors gave the Korovkin theorem on modular spaces including as particular cases L_p, Orlicz and Musielak-Orlicz spaces. After their work, many authors studied Korovkin type theorems on modular spaces (see [4, 7, 16, 17, 28, 29]). Recently, Yılmaz et al. [36] defined a new type of modular convergence by using the notion of relative uniform convergence ([9, 23]). Then, Korovkin type theorems have been studied via the statistical version of this new notion by Demirci and Kolay ([10]) for single sequences and by Demirci and Orhan ([12]) for double sequences. Also, Korovkin-type approximation theorems are studied via $\mathcal{A}-$summation process in modular spaces (see [13, 18, 28, 30]). In this work, we obtain the abstract versions of the Korovkin type approximation theorems via statistical relative $\mathcal{A}-$Summation process in modular spaces for double sequences of positive linear operators. Hence, we changed classical test functions of Korovkin theorem. In Section 2, we introduce the notations and definitions which are needed and states the results. The proofs of the main results are given in Section 3. Section 4 gives an application showing that our results are stronger. Finally, in the last section, we show that the positivity condition of linear operators in the Korovkin theorems can be relaxed.

2 Preliminaries

For the purposes of the present paper, we begin by recalling the concept of Pringsheim convergence.

A double sequence $x = (x_{mn})$ is convergent to L in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > N$ and denoted by $P-\lim_{m,n} x_{mn} = L$

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(see [33]). A double sequence is bounded if there exists a positive number B such that $|x_{mn}| \leq B$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. We note that contrary to the case for single sequences, a convergent double sequence need not to be bounded.

The concept of statistical convergence for double sequences was introduced and studied by Moricz [24] and can be reformulated in terms of natural density.

Let $S \subset \mathbb{N}^2$ be a two-dimensional subset of positive integers and let $S_{mn} = \{(k,l) \in S : k \leq m,\ l \leq n\}$. Then the two-dimensional analogue of natural density can be defined as follows:

$$\delta_2(S) := P - \lim_{m,n} \frac{1}{mn} |S_{mn}|$$

if it exists. The number sequence $x = (x_{mn})$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set $S := S_{mn}(\varepsilon) := \{k \leq m,\ l \leq n : \ |x_{kl} - L| \geq \varepsilon\}$ has natural density zero; in that case we write $st_2 - \lim_{m,n} x_{mn} = L$. Also, this convergence method was characterized in [24] as given below:

A double sequence $x = (x_{mn})$ is statistically convergent to L if and only if there exists a set $S \subset \mathbb{N}^2$ such that the natural density of S is 1 and

$$P - \lim_{\substack{m,n\to\infty \\ \text{and } (m,n)\in S}} x_{mn} = L.$$

Clearly, a $P-$convergent double sequence is statistically convergent to the same value but its converse is not always true.

Now we recall the concepts of statistical superior limit and inferior limit for double sequences have been introduced by Çakan and Altay [8]. For any real double sequence $x = (x_{mn})$, *the statistical limit superior of* x is

$$st_2 - \limsup_{m,n} x_{mn} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \{C \in \mathbb{R} : \delta_2 (\{(m,n) : x_{mn} > C\}) \neq 0\}$ and $\varnothing$ denotes the empty set. It means in general to be $\delta_2 (G_x) \neq 0$ that either $\delta_2 (G_x) > 0$ or G_x fails to have the double natural density. Similarly, the statistical limit inferior of x is

$$st_2 - \liminf_{m,n} x_{mn} = \begin{cases} \inf F_x, & \text{if } F_x \neq \varnothing, \\ \infty, & \text{if } F_x = \varnothing, \end{cases}$$

where $F_x := \{D \in \mathbb{R} : \delta_2 (\{(m,n) : x_{mn} < D\}) \neq 0\}$. The ordering relation between these concepts is similiar as in the ordinary superior or inferior limit, i.e.,

$$st_2 - \liminf_{m,n} x_{mn} \leq st_2 - \limsup_{m,n} x_{mn}.$$

We recall some notations related to the summability theory.

Let $A = [a_{klmn}]$, $k,l,m,n \in \mathbb{N}$, be a four-dimensional infinite matrix. For a given double sequence $x = (x_{mn})$, the $A-$transform of x, denoted by $Ax := ((Ax)_{kl})$, is given by

$$(Ax)_{kl} = \sum_{(m,n)\in\mathbb{N}^2} a_{klmn} x_{mn}, \quad k,l \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$. We say that a sequence x is $A-$summable to L if the $A-$transform of x exists for all $k,l\in\mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P-\lim_{p,q}\sum_{m=1}^{p}\sum_{n=1}^{q}a_{klmn}x_{mn}=y_{kl} \text{ and } P-\lim_{k,l}y_{kl}=L.$$

In summability theory, a two-dimensional matrix transformation is called regular if it maps every convergent sequence in to a convergent sequence with the same limit.

Now let $\mathcal{A}:=\left(A^{(i,j)}\right)=\left(a_{klmn}^{(i,j)}\right)$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x=(x_{mn})$ is said to be $\mathcal{A}-$*summable* to L if

$$P-\lim_{k,l}\sum_{(m,n)\in\mathbb{N}^2}a_{klmn}^{(i,j)}x_{mn}=L$$

uniformly in i and j.

If $A^{(i,j)}=A$, four-dimensional infinite matrix, then $\mathcal{A}-$summability is the $A-$summability for four-dimensional infinite matrix. Some results regarding matrix summability method for double sequences may be found in the papers [32], [35].

Now, we start by giving basic concepts and facts of modular spaces.

Assume that X be a locally compact Hausdorff topological space with a uniform structure $\mathcal{U}\subset 2^{X\times X}$ that generates the topology of X (see, [21]). Let $\mathcal{B}$ be the $\sigma-$algebra of all Borel subsets of X and $\mu:\mathcal{B}\to\mathbb{R}$ is a positive $\sigma-$finite regular measure. Let $L^0(X)$ be the space of all real valued $\mu-$measurable functions on X provided with equality almost everywhere, $C_b(X)$ be the space of all continuous real valued and bounded functions on X and $C_c(X)$ be the subspace of $C_b(X)$ of all functions with compact support on X. In this case, we say that a functional $\rho:L^0(X)\to[0,\infty]$ is a modular on $L^0(X)$ if it satisfies the following conditions:

(i) $\rho(f)=0$ if and only if $f=0$ $\mu-$almost everywhere on X,

(ii) $\rho(-f)=\rho(f)$ for every $f\in L^0(X)$,

(iii) $\rho(\alpha f+\beta g)\le\rho(f)+\rho(g)$ for every $f,g\in L^0(X)$ and for any $\alpha,\beta\ge 0$ with $\alpha+\beta=1$.

A modular ρ is $N-$quasi convex if there exists a constant $N\ge 1$ such that the inequality $\rho(\alpha f+\beta g)\le N\alpha\rho(Nf)+N\beta\rho(Ng)$ holds for every $f,g\in L^0(X)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$. Note that if $N=1$, then ρ is called convex. Furthermore, a modular ρ is $N-$quasi semiconvex if there exists a constant $N\ge 1$ such that $\rho(\alpha f)\le N\alpha\rho(Nf)$ holds for every $f\in L^0(X)$ and $\alpha\in(0,1]$.

The modular space X_ρ generated by modular ρ, given by

$$X_\rho:=\left\{f\in L^0(X):\lim_{\lambda\to 0^+}\rho(\lambda f)=0\right\}$$

and the space of the finite elements of X_ρ, given by

$$X_\rho^*:=\{f\in X_\rho:\rho(\lambda f)<\infty \text{ for all } \lambda>0\}.$$

Also, note that if ρ is $N-$quasi semiconvex, then the space

$\{f \in L^0(X) : \rho(\lambda f) < \infty$ for some $\lambda > 0\}$ coincides with X_ρ.

Now we recall the statistical relative modular and strong convergence for double sequences (see also [12]).

Definition 2.1. Let (f_{mn}) be a double function sequence whose terms belong to X_ρ. Then, (f_{mn}) is said to be *statistically relatively modularly convergent* to a function $f \in X_\rho$ if there exists a function $\sigma(u)$, called a scale function $\sigma \in L^0(X)$, $|\sigma(u)| \neq 0$ such that

$$st_2 - \lim_{m,n} \rho\left(\lambda_0 \left(\frac{f_{mn} - f}{\sigma}\right)\right) = 0 \ \text{ for some } \lambda_0 > 0.$$

Also, (f_{mn}) is *statistically relatively $F-$norm convergent* (or, *statistically relatively strongly convergent*) to f iff

$$st_2 - \lim_{m,n} \rho\left(\lambda \left(\frac{f_{mn} - f}{\sigma}\right)\right) = 0 \ \text{ for every } \lambda > 0.$$

The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in L^0(X)$, see [26].

Note that if the scale function is selected a non-zero constant, then statistical modular convergence is the special case of statistical relative modular convergence. Moreover, if $\sigma(u)$ is bounded, statistical relative modular convergence implies statistical modular convergence. However, if $\sigma(u)$ is unbounded, then statistical relative modular convergence does not imply statistical modular convergence.

Recently, Orhan and Kolay ([30]) presented *$\mathcal{A}-$summation process* for double sequences on a modular space and more recently, Demirci, Orhan and Kolay ([13]) introduced the notion of *relative modular $\mathcal{A}-$summation process* for double sequences as follows:

A sequence $\mathbb{T} := (T_{mn})$ of positive linear operators of D into $L^0(X)$ is called a *relative $\mathcal{A}-$summation process* on D if $(T_{mn}f)$ is relatively $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$P - \lim_{k,l} \rho\left[\lambda\left(\frac{A^{\mathbb{T}}_{klij} f - f}{\sigma}\right)\right] = 0, \text{ uniformly in } i, j, \text{ for some } \lambda > 0,$$

where for all $k, l, i, j \in \mathbb{N}$, $f \in D$ the series

$$A^{\mathbb{T}}_{klij} f := \sum_{(m,n) \in \mathbb{N}^2} a^{(i,j)}_{klmn} T_{mn} f$$

are absolutely convergent almost everywhere with respect to Lebesgue measure and we denote the value of $T_{mn}f$ at a point $u \in X$ by $T_{mn}(f(v); u)$ or briefly, $T_{mn}(f; u)$. It will be observed that *$\mathcal{A}-$summation process* is the special case of *relative $\mathcal{A}-$summation process* in which the scale function is a non-zero constant.

In this regard, some results on this new convergence method can be obtained by applying some Korovkin type theorems for double sequences of linear operators on a modular space.

In the present paper, we consider the following assumptions:

- ♦ A modular ρ is said to be monotone if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$.

- A modular ρ is finite if $\chi_A \in X_\rho$ whenever $A \in \mathcal{B}$ with $\mu(A) < \infty$.
- A modular ρ is strongly finite if $\chi_A \in X_\rho^*$ for all $A \in \mathcal{B}$ such that $\mu(A) < \infty$.
- A modular ρ is said to be absolutely continuous if there exists an $\alpha > 0$ such that, for every $f \in L^0(X)$ with $\rho(f) < \infty$, the following conditions hold:

∘ for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ such that $\mu(A) < \infty$ and $\rho\left(\alpha f \chi_{X \setminus A}\right) \leq \varepsilon$,

∘ for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho(\alpha f \chi_B) \leq \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

If a modular ρ is monotone and finite, then $C(X) \subset X_\rho$. If ρ is monotone and strongly finite, then $C(X) \subset X_\rho^*$. Also, if ρ is monotone, strongly finite and absolutely continuous, $\overline{C_c(X)} = X_\rho$ with respect to the modular convergence in the ordinary sense (see [20, 22, 27]).

3 The main results

We now prove some Korovkin type theorems with respect to an abstract finite set of test functions $f_0, f_1, ..., f_q$ in the sense of statistical relative $\mathcal{A}-$Summation process in modular spaces.

Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators from D into $L^0(X)$ with $C_b(X) \subset D \subset L^0(X)$. Let ρ be monotone and finite modular on $L^0(X)$. Assume further that the double sequence $\mathbb{T}$, together with modular ρ, satisfies the following property:

there exists a subset $X_\mathbb{T} \subset D \cap X_\rho$ with $C_b(X) \subset X_\mathbb{T}$ and $\sigma \in L^0(X)$ is an unbounded function satisfying $\sigma(u) \neq 0$ such that the inequality

$$st_2 - \limsup_{k,l} \rho\left(\lambda\left(\frac{A_{klij}^{\mathbb{T}} h}{\sigma}\right)\right) \leq R\rho(\lambda h), \text{ uniformly in } i, j, \tag{3.1}$$

holds for every $h \in X_\mathbb{T}$, $\lambda > 0$ and for an absolute positive constant R.

Set $f_0(v) \equiv 1$ for all $v \in X$, let f_r, $r = 1, 2, ..., q$ and a_r, $r = 0, 1, 2, ..., q$, be functions in $C_b(X)$. Put

$$P_u(v) = \sum_{r=0}^{q} a_r(u) f_r(v), \ u, v \in X, \tag{3.2}$$

and suppose that $P_u(v)$, $u, v \in X$, satisfies the following properties:

$(P1)$ $P_u(u) = 0$, for all $u \in X$,

$(P2)$ for every neighbourhood $U \in \mathcal{U}$ there is a positive real number η with $P_u(v) \geq \eta$ whenever $u, v \in X$, $(u, v) \notin U$ (see for examples [4]).

In order to obtain our main theorem, we first give the following result.

Theorem 3.1. Let $\mathcal{A} = \left(A^{(i,j)}\right)$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite and $N-$quasi semiconvex modular. Suppose that f_r and a_r, $r = 0, 1, 2, ..., q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators from D into $L^0(X)$ and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)| \geq b_r > 0$ $(r = 0, 1, 2, ..., q)$. If

$$st_2 - \lim_{k,l} \rho\left(\lambda_0\left(\frac{A^{\mathbb{T}}_{klij} f_r - f_r}{\sigma_r}\right)\right) = 0, \text{ uniformly in } i, j, \tag{3.3}$$

for some $\lambda_0 > 0$, $r = 0, 1, 2, ..., q$, in X_ρ then for every $f \in C_c(X)$

$$st_2 - \lim_{k,l} \rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij} f - f}{\sigma}\right)\right) = 0, \text{ uniformly in } i, j, \tag{3.4}$$

for some $\gamma > 0$, in X_ρ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., q\}$. If

$$st_2 - \lim_{k,l} \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{klij} f_r - f_r}{\sigma_r}\right)\right) = 0, \text{ uniformly in } i, j,$$

for every $\lambda > 0$, $r = 0, 1, 2, ..., q$, in X_ρ then for every $f \in C_c(X)$

$$st_2 - \lim_{k,l} \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{klij} f - f}{\sigma}\right)\right) = 0, \text{ uniformly in } i, j,$$

for every $\lambda > 0$, in X_ρ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., q\}$.

Proof. We first claim that, for every $f \in C_c(X)$,

$$st_2 - \lim_{k,l} \rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij} f - f}{\sigma}\right)\right) = 0, \text{ uniformly in } i, j, \tag{3.5}$$

for some $\gamma > 0$. To see this, assume that $f \in C_c(X)$. Then, since X is endowed with the uniformity $\mathcal{U}$, f is uniformly continuous and bounded on X. By the uniform continuity of f, choose $\varepsilon \in (0, 1]$, there exists a set $U \in \mathcal{U}$ such that $|f(u) - f(v)| \leq \varepsilon$ whenever $u, v \in X$, $(u, v) \in U$.

For all $u, v \in X$ let $P_u(v)$ be as in (3.2), and $\eta > 0$ satisfy condition $(P2)$. Then for $u, v \in X$, $(u, v) \notin U$, we have $|f(u) - f(v)| \leq \frac{2M}{\eta} P_u(v)$ where $M := \sup_{v \in X} |f(v)|$. Therefore, in any case we get $|f(u) - f(v)| \leq \varepsilon + \frac{2M}{\eta} P_u(v)$ for all $u, v \in X$, namely,

$$-\varepsilon - \frac{2M}{\eta} P_u(v) \leq f(u) - f(v) \leq \varepsilon + \frac{2M}{\eta} P_u(v). \tag{3.6}$$

Since T_{mn} is linear and positive, by applying T_{mn} to (3.6) for every $m, n \in \mathbb{N}$ we have

$$\begin{aligned} -\varepsilon A^{\mathbb{T}}_{klij}(f_0; u) - \frac{2M}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) &\leq f(u) A^{\mathbb{T}}_{klij}(f_0; u) - A^{\mathbb{T}}_{klij}(f; u) \\ &\leq \varepsilon A^{\mathbb{T}}_{klij}(f_0; u) + \frac{2M}{\eta} A^{\mathbb{T}}_{klij}(P_u; u). \end{aligned}$$

Hence

$$\begin{aligned} \left|A_{klij}^{\mathbb{T}}(f;u)-f(u)\right| &\leq \left|A_{klij}^{\mathbb{T}}(f;u)-f(u)A_{klij}^{\mathbb{T}}(f_0;u)\right| \\ &\quad+|f(u)|\left|A_{klij}^{\mathbb{T}}(f_0;u)-f_0(u)\right| \\ &\leq \varepsilon A_{klij}^{\mathbb{T}}(f_0;u)+\frac{2M}{\eta}A_{klij}^{\mathbb{T}}(P_u;u)+M\left|A_{klij}^{\mathbb{T}}(f_0;u)-f_0(u)\right| \\ &\leq \varepsilon+(\varepsilon+M)\left|A_{klij}^{\mathbb{T}}(f_0;u)-f_0(u)\right| \\ &\quad+\frac{2M}{\eta}\sum_{r=0}^{q}a_r(u)\left|A_{klij}^{\mathbb{T}}(f_r;u)-f_r(u)\right|. \end{aligned}$$

Let $\gamma>0$. Now for each $r=0,1,2,...,q$ and $u\in X$, choose $M_0>0$ such that $|a_r(u)|\leq M_0$ and multiplying the both sides of the above inequality by $\frac{1}{|\sigma(u)|}$, the last inequality gives that

$$\gamma\left|\frac{A_{klij}^{\mathbb{T}}(f;u)-f(u)}{\sigma(u)}\right|\leq\frac{\gamma\varepsilon}{|\sigma(u)|}+K\gamma\sum_{r=0}^{q}\left|\frac{A_{klij}^{\mathbb{T}}(f_r;u)-f_r(u)}{\sigma(u)}\right|$$

where $K:=\varepsilon+M+\frac{2M}{\eta}M_0$. Now, applying the modular ρ to both sides of the above inequality, since ρ is monotone and $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,q\}$, we get

$$\rho\left(\gamma\left(\frac{A_{klij}^{\mathbb{T}}f-f}{\sigma}\right)\right)\leq\rho\left(\frac{\gamma\varepsilon}{|\sigma|}+K\gamma\sum_{r=0}^{q}\frac{A_{klij}^{\mathbb{T}}f_r-f_r}{\sigma_r}\right).$$

Thus, we can see that

$$\rho\left(\gamma\left(\frac{A_{klij}^{\mathbb{T}}f-f}{\sigma}\right)\right)\leq\rho\left(\frac{(q+2)\gamma\varepsilon}{\sigma}\right)+\sum_{r=0}^{q}\rho\left((q+2)K\gamma\left(\frac{A_{klij}^{\mathbb{T}}f_r-f_r}{\sigma_r}\right)\right).$$

Since ρ is $N-$quasi semiconvex and strongly finite, we have,

$$\rho\left(\gamma\left(\frac{A_{klij}^{\mathbb{T}}f-f}{\sigma}\right)\right)\leq N\varepsilon\rho\left(\frac{(q+2)\gamma N}{\sigma}\right)+\sum_{r=0}^{q}\rho\left((q+2)K\gamma\left(\frac{A_{klij}^{\mathbb{T}}f_r-f_r}{\sigma_r}\right)\right). \tag{3.7}$$

For a given $\varepsilon^*>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{(q+2)\gamma N}{\sigma}\right)<\varepsilon^*$. Now define the following sets:

$$\begin{aligned} S_\gamma &:= \left\{(k,l):\rho\left(\gamma\left(\frac{A_{klij}^{\mathbb{T}}f-f}{\sigma}\right)\right)\geq\varepsilon^*\right\} \\ S_{\gamma,r} &:= \left\{(k,l):\rho\left((q+2)K\gamma\left(\frac{A_{klij}^{\mathbb{T}}f_r-f_r}{\sigma_r}\right)\right)\geq\frac{\varepsilon^*-N\varepsilon\rho\left(\frac{(q+2)\gamma N}{\sigma}\right)}{q+1}\right\}, \end{aligned}$$

where $r=0,1,2,...,q$. Then, it is easy to see that $S_\gamma\subseteq\bigcup_{r=0}^{q}S_{\gamma,r}$. Hence, we have

$$\delta_2(S_\gamma)\leq\sum_{r=0}^{q}\delta_2(S_{\gamma,r}).$$

Using the hypothesis (3.3), we get

$$\delta_2 (S_\gamma) = 0,$$

which proves our claim (3.5).

The last part of theorem can be proved similarly to the first one. ∎

Now, we can give our main theorem of this paper.

Theorem 3.2. Let $\mathcal{A} = \left(A^{(i,j)}\right)$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that f_r and a_r, $r = 0, 1, 2, ..., q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators satisfying (3.1) and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)| \geq b_r > 0$ $(r = 0, 1, 2, ..., q)$. If

$$st_2 - \lim_{k,l} \rho \left(\lambda \left(\frac{A^{\mathbb{T}}_{klij} f_r - f_r}{\sigma_r} \right) \right) = 0, \text{ uniformly in } i, j,$$

for every $\lambda > 0$, $r = 0, 1, 2, ..., q$, in X_ρ, then for every $f \in D \cap X_\rho$ with $f - C_b(X) \subset X_{\mathbb{T}}$,

$$st_2 - \lim_{k,l} \rho \left(\lambda_0 \left(\frac{A^{\mathbb{T}}_{klij} f - f}{\sigma} \right) \right) = 0, \text{ uniformly in } i, j,$$

for some $\lambda_0 > 0$, in X_ρ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., q\}$ and D, $X_{\mathbb{T}}$ are as before.

Proof. Let $f \in D \cap X_\rho$ with $f - C_b(X) \subset X_{\mathbb{T}}$. It is known from [6, 22] that there exists a sequence $(g_{kl}) \subset C_c(X)$ such that $\rho(3\lambda_0^* f) < \infty$ and $P - \lim_{k,l} \rho (3\lambda_0^* (g_{kl} - f)) = 0$ for some $\lambda_0^* > 0$. This means that, for every $\varepsilon > 0$, there is a positive number $l_0 = l_0(\varepsilon)$ with

$$\rho (3\lambda_0^* (g_{kl} - f)) < \varepsilon \text{ for every } k, l \geq l_0. \tag{3.8}$$

For all $m, n \in \mathbb{N}$, by linearity and positivity of the operators T_{mn}, we have

$$\begin{aligned} & \lambda_0^* \left| A^{\mathbb{T}}_{klij} (f; u) - f(u) \right| \\ \leq \; & \lambda_0^* \left| A^{\mathbb{T}}_{klij} (f - g_{l_0 l_0}; u) \right| + \lambda_0^* \left| A^{\mathbb{T}}_{klij} (g_{l_0 l_0}; u) - g_{l_0 l_0}(u) \right| \\ & + \lambda_0^* \left| g_{l_0 l_0}(u) - f(u) \right| \end{aligned}$$

holds for every $u \in X$. Now, applying modular ρ in the last inequality and using the monotonicity of ρ and moreover multiplying the both sides of the above inequality by $\frac{1}{|\sigma(u)|}$, we get

$$\begin{aligned} & \rho \left(\lambda_0^* \left(\frac{A^{\mathbb{T}}_{klij} f - f}{\sigma} \right) \right) \\ \leq \; & \rho \left(3\lambda_0^* \left(\frac{A^{\mathbb{T}}_{klij} (f - g_{l_0 l_0})}{\sigma} \right) \right) + \rho \left(3\lambda_0^* \left(\frac{A^{\mathbb{T}}_{klij} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma} \right) \right) \\ & + \rho \left(3\lambda_0^* \left(\frac{g_{l_0 l_0} - f}{\sigma} \right) \right). \end{aligned}$$

Hence, observing $|\sigma| \geq b > 0$, $(b = \max\{b_r : r = 0, 1, 2, ..., q\})$, we can write that

$$\begin{aligned} &\rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} f - f}{\sigma}\right)\right) \\ \leq\ &\rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}}\left(f - g_{l_0 l_0}\right)}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0,l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ &+\rho\left(\frac{3\lambda_0^*}{b}\left(g_{l_0 l_0} - f\right)\right). \end{aligned} \tag{3.9}$$

Then using the (3.8) in (3.9), we have

$$\begin{aligned} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} f - f}{\sigma}\right)\right) \leq\ &\varepsilon + \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}}\left(f - g_{l_0 l_0}\right)}{\sigma}\right)\right) \\ &+\rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right). \end{aligned}$$

By property (3.1) and also using the facts that $g_{l_0 l_0} \in C_c(G)$ and $f - g_{l_0 l_0} \in X_{\mathbb{T}}$, we obtain

$$\begin{aligned} &st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} f - f}{\sigma}\right)\right) \\ \leq\ &\varepsilon + R\rho\left(3\lambda_0^*\left(f - g_{l_0 l_0}\right)\right) + st_2 - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ \leq\ &\varepsilon\left(1 + R\right) + st_2 - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \end{aligned}$$

also, resulting from previous theorem,

$$\begin{aligned} 0 &= st_2 - \lim_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ &= st_2 - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \end{aligned}$$

which gives

$$0 \leq st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} f - f}{\sigma}\right)\right) \leq \varepsilon\left(1 + R\right).$$

From arbitrariness of $\varepsilon > 0$, it follows that

$$st_2 - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} f - f}{\sigma}\right)\right) = 0.$$

Furthermore,

$$st_2 - \lim_{k,l} \rho \left(\lambda_0^* \left(\frac{A_{klij}^{\mathbb{T}} f - f}{\sigma} \right) \right) = 0,$$

this completes the proof. ∎

Remark 3.3. Note that, in Theorem 3.2, in general it is not possible to obtain statistical relative strong convergence unless the modular ρ satisfies the Δ_2-condition.

If one replaces the scale function by a nonzero constant, then the condition (3.1) reduces to

$$st_2 - \limsup_{k,l} \rho \left(\lambda \left(A_{klij}^{\mathbb{T}} h \right) \right) \leq R\rho (\lambda h), \text{ uniformly in } i,j, \tag{3.10}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the next result immediately follows from our Theorem 3.2.

Corollary 3.4. Let $\mathcal{A} = \left(A^{(i,j)} \right)$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that f_r and a_r, $r = 0, 1, 2, ..., q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators satisfying (3.10). If $\left(A_{klij}^{\mathbb{T}} f_r \right)$ is statistically strongly convergent to f_r, $r = 0, 1, 2, ..., q$, , uniformly in i, j, in X_ρ, then $\left(A_{klij}^{\mathbb{T}} f \right)$ is statistically modularly convergent to f , uniformly in i, j, in X_ρ such that f is any function belonging to $D \cap X_\rho$ with $f - C_b (X) \subset X_{\mathbb{T}}$.

If one replaces the matrices $A^{(i,j)}$ by the identity matrix and take the scale function as a non-zero constant, then the condition (3.1) reduces to

$$st_2 - \limsup_{k,l} \rho (\lambda (T_{kl} h)) \leq R\rho (\lambda h) \tag{3.11}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the following result immediately follows from our Theorem 3.2.

Corollary 3.5. Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that f_r and a_r, $r = 0, 1, 2, ..., q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators satisfying (3.11) and assume that $\sigma_r (u)$ is an unbounded function satisfying $|\sigma_r (u)| \geq b_r > 0$ $(r = 0, 1, 2, ..., q)$. If $(T_{mn} f_r)$ is statistically strongly convergent to f_r *to the scale function* σ_r, $r = 0, 1, 2, ..., q$, in X_ρ then $(T_{mn} f)$ is statistically modularly convergent to f *to the scale function* σ in X_ρ where $\sigma (u) = \max \{ |\sigma_r (u)| : r = 0, 1, 2, ..., q \}$ and f is any function belonging to $D \cap X_\rho$ with $f - C_b (X) \subset X_{\mathbb{T}}$.

4 Application

Now, we give an application showing that in general, our results are stronger than classical ones.

Example 4.1. Let us consider $X = [0,1]^2 = [0,1] \times [0,1] \subset \mathbb{R}^2$ and let $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function with φ is convex, $\varphi (0) = 0$, $\varphi (x) > 0$ *for any* $x > 0$ *and* $\lim_{x \to \infty} \varphi (x) = \infty$.

Then, the functional ρ^{φ} defined by

$$\rho^{\varphi}(f) := \int_0^1 \int_0^1 \varphi\left(|f(x,y)|\right) dxdy \quad \text{for} \quad f \in L^0(X),$$

is a convex modular on $L^0(X)$ and

$$X_{\rho^{\varphi}} := \left\{ f \in L^0(X) : \rho^{\varphi}(\lambda f) < +\infty \quad \text{for some } \lambda > 0 \right\}$$

is the Orlicz space generated by φ.

For every $(x,y) \in X$, let $f_0(x,y) = a_3(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$, $f_3(x,y) = a_0(x,y) = x^2 + y^2$, $a_1(x,y) = -2x$, $a_2(x,y) = -2y$. For every $m,n \in \mathbb{N}$, $u_1, u_2 \in [0,1]$, let $K_{mn}(u_1,u_2) = (m+1)(n+1) u_1^m u_2^n$ and for $f \in C(X)$ and $(x,y) \in X$, set

$$M_{mn}(f;x,y) = \int_0^1 \int_0^1 K_{mn}(u_1,u_2) f(u_1 x, u_2 y) du_1 du_2.$$

Then we get

$$\begin{aligned} & \int_0^1 \int_0^1 K_{mn}(u_1,u_2) du_1 du_2 \\ = \quad & (m+1)\left(\int_0^1 u_1^m du_1\right)(n+1)\left(\int_0^1 u_2^n du_2\right) = 1, \end{aligned}$$

and hence, $M_{mn}(f_0;x,y) = f_0(x,y) = 1$. Also, we know from [3] that

$$\begin{aligned} |M_{mn}(f_1;x,y) - f_1(x,y)| &\leq \frac{1}{m+2}, \ |M_{mn}(f_2;x,y) - f_2(x,y)| \leq \frac{1}{n+2}, \\ \left|M_{mn}\left(f_1^2;x,y\right) - f_1^2(x,y)\right| &\leq \frac{2}{m+3}, \ \left|M_{mn}\left(f_2^2;x,y\right) - f_2^2(x,y)\right| \leq \frac{2}{n+3}, \end{aligned}$$

and for each $m,n \geq 2$, $f \in X_{\rho^{\varphi}}$ we get $\rho^{\varphi}(M_{mn}f) \leq 32\rho^{\varphi}(f)$. Moreover, (M_{mn}) satisfies the condition (14) in [29] with $X_{\mathbb{M}} = X_{\rho^{\varphi}}$ and $(M_{mn}f)$ is modulary convergent to $f \in X_{\rho^{\varphi}}$. Using the operators $\mathbb{M} = (M_{mn})$, we define the double sequence of positive linear operators $\mathbb{L} = (L_{mn})$ on $X_{\rho^{\varphi}}$ as follows:

$$L_{mn}(f;x,y) = (1 + g_{mn}(x,y)) M_{mn}(f;x,y), \text{ for } f \in X_{\rho^{\varphi}},$$

$x,y \in [0,1]$ and $m,n \in \mathbb{N}$, where $g_{mn} : X \to \mathbb{R}$ defined by

$$\begin{aligned} & g_{mn}(x,y) \\ = \quad & \begin{cases} 1, & m = s^2 \text{ and } n = t^2 \\ mn(1 - mnxy), & (x,y) \in \left(0, \frac{1}{m}\right) \times \left(0, \frac{1}{n}\right); m \neq s^2 \text{ and } n \neq t^2 \\ 0, & (x,y) \notin \left(0, \frac{1}{m}\right) \times \left(0, \frac{1}{n}\right);\ m \neq s^2 \text{ and } n \neq t^2 \end{cases} \end{aligned}$$

$s,t=1,2,...$ If $\varphi(x)=x^p$ for $1\le p<\infty$, $x\ge 0$ then $X_{\rho^\varphi}=L_p(X)$ and we have for any function $f\in X_{\rho^\varphi}$, $\rho^\varphi(f)=\|f\|_p^p$. Choose $p=1$.

It is clear that

$$\begin{aligned}\rho\left(\lambda_0\left(g_{mn}-g\right)\right) &= \left\|\lambda_0\left(g_{mn}-g\right)\right\|_1\\ &= \lambda_0\begin{cases}1, & m=s^2 \text{ and } n=t^2\\ \frac{3}{4}, & (x,y)\in\left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right); m\neq s^2 \text{ and } n\neq t^2\\ 0, & (x,y)\notin\left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right);\ m\neq s^2 \text{ and } n\neq t^2\end{cases},\end{aligned}$$

$s,t=1,2,...,$ where $g=0$, then (g_{mn}) does not converge statistically modularly to $g=0$. Now, we choose $\sigma_r(x,y)=\sigma(x,y)$ $(r=0,1,2,3)$ where $\sigma(x,y)=\begin{cases}\frac{1}{xy}, & (x,y)\in(0,1]\times(0,1]\\ 1, & \text{otherwise}\end{cases}$ *on* $L_1(X)$. Then, we get

$$\begin{aligned}\rho\left(\lambda_0\left(\frac{g_{mn}-g}{\sigma}\right)\right) &= \left\|\lambda_0\left(\frac{g_{mn}-g}{\sigma}\right)\right\|_1\\ &= \lambda_0\begin{cases}1, & m=s^2 \text{ and } n=t^2\\ \frac{5}{36mn}, & (x,y)\in\left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right); m\neq s^2 \text{ and } n\neq t^2\\ 0, & (x,y)\notin\left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right);\ m\neq s^2 \text{ and } n\neq t^2\end{cases},\end{aligned}$$

and (g_{mn}) converges statistically modularly to $g=0$ *to the scale function* σ. Also, *assume that* $\mathcal{A}:=\left(A^{(i,j)}\right)=\left(a_{klmn}^{(i,j)}\right)$ *is a sequence of four dimensional infinite matrices defined by* $a_{klmn}^{(i,j)}=\frac{1}{kl}$ *if* $i\le m\le i+k-1$, $j\le n\le j+l-1$, $(i,j=1,2,...)$ *and* $a_{klmn}^{(i,j)}=0$ *otherwise. In this case* $\mathcal{A}-$*summability method reduces to almost* convergence of double sequences introduced by Moricz and Rhoades *[25].* Then, it can be seen that, for every $L_1(X)$, $\lambda>0$ and for positive constant R_0 that

$$st_2-\limsup_{k,l}\left\|\lambda\left(\frac{A_{klij}^{\mathbb{L}}h}{\sigma}\right)\right\|_1\le R_0\left\|\lambda h\right\|_1,\ \text{uniformly in } i,j.$$

Now, observe that

$$\begin{aligned}L_{mn}\left(f_0;x,y\right)-f_0\left(x,y\right) &= g_{mn}\left(x,y\right),\\ L_{mn}\left(f_1;x,y\right)-f_1\left(x,y\right) &\le \frac{1+g_{mn}\left(x,y\right)}{m+2}+g_{mn}\left(x,y\right),\\ L_{mn}\left(f_2;x,y\right)-f_2\left(x,y\right) &\le \frac{1+g_{mn}\left(x,y\right)}{n+2}+g_{mn}\left(x,y\right),\\ L_{mn}\left(f_3;x,y\right)-f_3\left(x,y\right) &\le \left(1+g_{mn}\left(x,y\right)\right)\left(\frac{2}{m+3}+\frac{2}{n+3}\right)+2g_{mn}\left(x,y\right).\end{aligned}$$

Hence, we can see, for any $\lambda > 0$, that

$$\begin{aligned}
\left\|\lambda\left(\frac{A^{\mathbb{L}}_{klij}f_0 - f_0}{\sigma}\right)\right\|_1 &= \left\|\frac{\lambda}{\sigma}\left(\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}(1+g_{mn}) - 1\right)\right\|_1 \\
&\leq \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\left\|\lambda\frac{g_{mn}}{\sigma}\right\|_1 \\
&= \lambda\begin{cases} 1, & m = s^2 \text{ and } n = t^2 \\ \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{5}{36mn}, & (x,y) \in \left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right); m \neq s^2 \text{ and } n \neq t^2 \\ 0, & (x,y) \notin \left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right);\ m \neq s^2 \text{ and } n \neq t^2 \end{cases}, \qquad (4.1)
\end{aligned}$$

$s,t = 1,2,...,$ *Since* $P - \lim_{k,l}\left(\sup_{i,j}\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{5}{36mn}\right) = 0$, then we can easily see that

$$st_2 - \lim_{k,l}\left\|\lambda\left(\frac{A^{\mathbb{L}}_{klij}f_0 - f_0}{\sigma}\right)\right\|_1 = 0, \text{ uniformly in } i,j.$$

Also, we have

$$\begin{aligned}
\left\|\lambda\left(\frac{A^{\mathbb{L}}_{klij}f_1 - f_1}{\sigma}\right)\right\|_1 &\leq \left\|\frac{\lambda}{\sigma}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{1}{kl}\left(\frac{1+g_{mn}}{m+2} + g_{mn}\right)\right\|_1 \\
&\leq \frac{\lambda}{4}\left(\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{1}{m+2}\right) \\
&\quad + \frac{2}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\left\|\lambda\frac{g_{mn}}{\sigma}\right\|_1.
\end{aligned}$$

Since $st_2 - \lim_{k,l}\left(\sup_{i,j}\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{1}{m+2}\right) = 0$ and from the inequality (4.1), we have

$$st_2 - \lim_{k,l}\left\|\lambda\left(\frac{A^{\mathbb{L}}_{klij}f_1 - f_1}{\sigma}\right)\right\|_1 = 0, \text{ uniformly in } i,j.$$

Similarly, we get

$$st_2 - \lim_{k,l}\left\|\lambda\left(\frac{A^{\mathbb{L}}_{klij}f_2 - f_2}{\sigma}\right)\right\|_1 = 0, \text{ uniformly in } i,j.$$

Finally, since

$$\begin{aligned} & \left\| \lambda \left(\frac{A_{klij}^{\mathbb{L}} f_3 - f_3}{\sigma} \right) \right\|_1 \\ \leq & \left\| \frac{\lambda}{\sigma} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \frac{1}{kl} \left((1 + g_{mn}(x,y)) \left(\frac{2}{m+3} + \frac{2}{n+3} \right) + 2g_{mn}(x,y) \right) \right\|_1 \\ \leq & \frac{\lambda}{4} \left(\frac{1}{kl} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \left(\frac{2}{m+3} + \frac{2}{n+3} \right) \right) + \frac{4}{kl} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \left\| \lambda \left(\frac{g_{mn}}{\sigma} \right) \right\|_1 . \end{aligned}$$

Hence we can easily see that

$$st_2 - \lim_{k,l} \left\| \lambda \left(\frac{A_{klij}^{\mathbb{L}} f_3 - f_3}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } i,j.$$

So, our new operator $\mathbb{L} = (L_{mn})$ satisfies all conditions of Theorem 3.2 and therefore we obtain

$$st_2 - \lim_{k,l} \left\| \lambda_0 \left(\frac{A_{klij}^{\mathbb{L}} f - f}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } i,j.$$

for some $\lambda_0 > 0$, for any $f \in L_1(X)$. However, $(L_{mn} f_0)$ is neither statistically $\mathcal{A}-$summable nor statistically modularly convergent to f_0. Thus (L_{mn}) does not fulfil the Corollary 3.4 and Corollary 3.5.

5 An extension to non-positive linear operators

In this section, we relax the positivity condition of linear operators in the Korovkin theorems. In [2, 3, 4] there are some positive answers. Following this approach, we give some positive answers also for statistical relative $\mathcal{A}-$Summation process in modular spaces and prove a Korovkin type approximation theorem.

Let G be a bounded interval of $\mathbb{R}^2$, $C^2(G)$ (resp. $C_b^2(G)$) be the space of all functions defined on G, (resp. bounded and) continuous together with their first and second derivatives, $C_+ := \left\{ f \in C_b^2(G) : f \geq 0 \right\}$, $C_+^2 := \left\{ f \in C_b^2(G) : f'' \geq 0 \right\}$.

Let f_r, $r = 1, 2, ..., q$, and a_r, $r = 0, 1, 2, ..., q$, be functions in $C_b^2(G)$, $P_u(v)$, $u, v \in G$, be as in (3.2), and suppose that $P_u(v)$ satisfies the properties $(P1)$, $(P2)$ and

$(P3)$ there is a positive real constant S_0 such that $P_u''(v) \geq S_0$ for all $u, v \in G$ (Here the second derivative is intended with respect to v).

Now we prove the following Korovkin type approximation theorem for not necessarily positive linear operators.

Theorem 5.1. Let $\mathcal{A}$, ρ and σ_r be as in Theorem 3.1 and f_r, a_r, $r = 0, 1, 2, ..., q$ and $P_u(v)$, $u, v \in G$, satisfies the properties $(P1)$, $(P2)$ and $(P3)$. Assume that $\mathbb{T} = (T_{mn})$ be a double sequence of linear operators and $T_{mn}\left(C_+ \cap C_+^2\right) \subset C_+$ for all $m, n \in \mathbb{N}$. If $A_{klij}^{\mathbb{T}} f_r$ is statistically relatively modularly convergent to f_r *to the scale function* σ_r in X_ρ for each $r = 0, 1, 2, ..., q$, then $A^{\mathbb{T}}_{klij} f$ is statistically relatively modularly convergent to f *to the scale function* σ in X_ρ for every $f \in C^2_b(G)$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., q\}$.

If $A^{\mathbb{T}}_{klij} f_r$ is statistically relatively strongly convergent to f_r *to the scale function* σ_r, $r = 0, 1, 2, ..., q$, in X_ρ then $A^{\mathbb{T}}_{klij} f$ is statistically relatively strongly convergent to f *to the scale function* σ in X_ρ for every $f \in C^2_b(G)$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., q\}$.

Furthermore, if ρ is absolutely continuous, $\mathbb{T}$ satisfies the property (3.1) and $A^{\mathbb{T}}_{klij} f_r$ is statistically relatively strongly convergent to f_r *to the scale function* σ_r, $r = 0, 1, 2, ..., q$, in X_ρ then $A^{\mathbb{T}}_{klij} f$ is statistically relatively modularly convergent to f *to the scale function* σ in X_ρ for every $f \in D \cap X_\rho$ with $f - C_b(G) \subset X_{\mathbb{T}}$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., q\}$.

Proof. Let $f \in C^2_b(G)$. Since f is uniformly continuous and bounded on G, given $\varepsilon > 0$ with $0 < \varepsilon \le 1$, there exists a $\delta > 0$ such that $|f(u) - f(v)| \le \varepsilon$ for all $u, v \in G$, $|u - v| \le \delta$. Let $P_u(v)$, $u, v \in G$, be as in (3.2) and let $\eta > 0$ be associated with δ, satisfying $(P2)$. As in Theorem 3.1, for every $\beta \ge 1$ and $u, v \in G$, we have

$$-\varepsilon - \frac{2M\beta}{\eta} P_u(v) \le f(u) - f(v) \le \varepsilon + \frac{2M\beta}{\eta} P_u(v) \tag{5.1}$$

where $M = \sup_{v \in G} |f(v)|$. From (5.1) it follows that

$$h_{1,\beta}(v) := \varepsilon + \frac{2M\beta}{\eta} P_u(v) + f(v) - f(u) \ge 0, \tag{5.2}$$

$$h_{2,\beta}(v) := \varepsilon + \frac{2M\beta}{\eta} P_u(v) - f(v) + f(u) \ge 0. \tag{5.3}$$

Let H_0 satisfy $(P3)$. For each $v \in G$, we get

$$h''_{1,\beta}(v) \ge \frac{2M\beta H_0}{\eta} + f''(v), \ h''_{2,\beta}(v) \ge \frac{2M\beta H_0}{\eta} - f''(v).$$

Because of f'' is bounded on G, we can choose $\beta \ge 1$ in such a way that $h''_{1,\beta}(v) \ge 0$, $h''_{2,\beta}(v) \ge 0$ for each $v \in G$. Hence $h_{1,\beta}, h_{2,\beta} \in C_+ \cap C^2_+$ and then, by hypothesis

$$A^{\mathbb{T}}_{klij}(h_{j,\beta}; u) \ge 0 \text{ for all } m, n \in \mathbb{N}, \ u \in G \text{ and } j = 1, 2. \tag{5.4}$$

From (5.2)-(5.4) and the linearity of T_{mn}, we get

$$\begin{aligned}
\varepsilon A^{\mathbb{T}}_{klij}(f_0; u) + \frac{2M\beta}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) + A^{\mathbb{T}}_{klij}(f; u) - f(u) A^{\mathbb{T}}_{klij}(f_0; u) &\ge 0, \\
\varepsilon A^{\mathbb{T}}_{klij}(f_0; u) + \frac{2M\beta}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) - A^{\mathbb{T}}_{klij}(f; u) + f(u) A^{\mathbb{T}}_{klij}(f_0; u) &\ge 0,
\end{aligned}$$

thus,

$$\begin{aligned}
-\varepsilon A^{\mathbb{T}}_{klij}(f_0; u) - \frac{2M\beta}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) &\le f(u) A^{\mathbb{T}}_{klij}(f_0; u) - A^{\mathbb{T}}_{klij}(f; u) \\
&\le \varepsilon A^{\mathbb{T}}_{klij}(f_0; u) + \frac{2M\beta}{\eta} A^{\mathbb{T}}_{klij}(P_u; u).
\end{aligned}$$

By arguing similarly as in the proof of Theorem 3.1, multiplying the inequality by $\frac{1}{|\sigma(u)|}$, using the modular ρ and for $k, l \in \mathbb{N}$, we have the assertion of the first part.

The other parts can be proved similarly as in the proofs of Theorem 3.1 and Theorem 3.2. ∎

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